A Counter-Intuitive Incompleteness Property of the Axiom System $I\Sigma_0$

by

Dan E. Willard

University at Albany

This talk will discuss both generalizations and boundary-case exceptions permitted by Gödel's Second Incompleteness Theorem. It will show that two logically equivalent axiomatizations for $I\Sigma_0$ have opposite incompleteness properties !

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3 Themes of This Talk:

1. Gödel's Incompleteness Theorem is an astonishingly powerful result.
2. Our Prior Research has found both generalizations and partial exceptions to Gödel's Second Incompleteness Theorem.
3. New Research will show that *Two Logically Equivalent* axiomatizations for $I\Sigma_0$ have fully opposite incompleteness properties!

   This surprising result will hold because our two "equivalent" axiomatizations, α and β, will prove identical sets of theorems *BUT NOT KNOW THAT* they prove identical sets of theorems !

Gödel's 1931 Paper Had Two Results:

FIRST INCOMPLETENESS THEOREM: No algorithm can list all True Statements of Arithmetic

SECOND INCOMPLETENESS THEOREM: No Axiom System of Conventional Strength Can Prove a Theorem Formally Confirming Its Own Self-Consistency.

Our JSL 2001 & 2005 Papers Explored:

Boundary-Case Exceptions to the Second Incompleteness Effect where an axiom system contains a formal axiom sentence stating:

"I am consistent" i.e. the union of the other axioms with **THIS STATEMENT** (looking at itself) is consistent.

MAIN RESULT OF JSL 2001 AND JSL 2005 was such Constructions are Reasonable Under Some Very Special and Tightly Controlled Circumstances.

Definition 1. A formula in the language or arithmetic (using the addition and multiplication symbols) *is called* Δ_0 iff all its quantifiers are bounded.

i.e. they look like $\forall\, x \leq t$ or $\exists\, x \leq t$

Definition 2. The axiom system IΣ_0 is defined to be an extension of Robinson's Axiom System Q that recognizes the validity of the Principle of Induction for Δ_0 formulae. Thus if $\phi(x, y)$ is Δ_0 then IΣ_0 contains the axiom:

$$\forall x \quad \{ \quad \{ \;\; \phi(x,0) \;\; \wedge \;\; \forall y\, [\, \phi(x,y) \Longrightarrow \phi(x,y+1) \,] \; \} \\ \Longrightarrow \;\; \forall y\; \phi(x,y) \quad \}$$

1981 Paris-Wilkie Open Question :

Does IΣ_0 satisfy the Herbrandized and semantic tableaux versions of the Gödel's Second Incompleteness Theorem?

Prior Literature has sometimes used term "IΔ_0" to refer to IΣ_0

Summary of Prior research :

1. Feferman (1960) warned us to *carefully separate* different definitions of consistency when generalizing Second Incomp Theorem.
2. Kriesel-Takeuti (1974) showed some logics could verify their cut-free consistency under a second-order logic generalization of sequent calculus
3. Wilkie-Paris 1987 showed $\mathrm{I}\Sigma_0 + Exp$ cannot prove Q's Hilbert consistency and asked whether $\mathrm{I}\Sigma_0$ could verify its Herbrandized and/or semantic tableaux consistency" ?
4. Adamowicz-Zbierski 2001 showed $\mathrm{I}\Sigma_0 + \Omega_1$ satisfies Herbrandized version of Second Incompleteness Theorem
5. Willard-2002 showed conventional axiomatization for $\mathrm{I}\Sigma_0$ satisfies the semantic tab version of Second Incompleteness Theorem.

Our New Result : *Unconventional axiomatizations for $I\Sigma_0$* Are Anti-Thresholds for Herbrandized Version of 2nd Incomp Theorem.

Although they are logically equivalent to its conventional axiomatizations !

Definition 3. Let $\phi(x, y)$ again denote a Δ_0 formula. There exists two logically equivalent axiomatizations for $I\Sigma_0$, called Ax-1 & Ax-2, based on the two different induction schemes below:

$$\forall x \quad \{ \quad \{ \quad \phi(x,0) \quad \wedge \quad \forall y \, [\, \phi(x,y) \Longrightarrow \phi(x,y+1) \,] \, \} \\ \Longrightarrow \quad \forall y \; \phi(x,y) \qquad \} \qquad\qquad (1)$$

$$\forall x \, \forall z \, \{ \, \{ \, \phi(x,0) \ \wedge \ \forall y \leq z \, [\, \phi(x,y) \Longrightarrow \phi(x,y+1) \,] \, \} \\ \Longrightarrow \quad \forall y \leq z \quad \phi(x,y) \qquad \} \qquad\qquad (2)$$

Kołodziejczyk's Email to Willard asked the following question:

How difficult would it be to generalize Willard's JSL-2002 article so that its generalization of the Second Incompleteness Theorem would extend to the Ax-2 formalism under Herbrand Deduction?

Surprising Answer to this Question:

While it is not difficult to generalize JSL-2002's methods to Ax-2, there exists a *third axiomatization for* $I\Sigma_0$, called Ax-3, which is an *anti-threshold* for the Herbrandized version of the Second Incompleteness Theorem.

Definition 4. The statement " $\alpha \supset \beta$ " means that the axiom system α contains all β's formal axioms.

Above is much stronger than the statement that " α can prove all β's theorems".

Definition 5. Let A denote a consistent axiom system and D denote a deduction method. Then (A, D) is an *Incompleteness Threshold* iff every consistent $\alpha \supset A$ is unable to prove the theorem statement that α is consistent under the deduction method D.

Definition 6. (A, D) is an *Anti-Threshold* when Definition 5's condition fails.

i.e. there exists a consistent $\alpha \supset A$ able to prove the theorem statement that α is consistent under deduction method D.

Main Surprising Result. One axiomatization for $\mathsf{I}\Sigma_0$ is a Herbrandized Threshold — *and oddly another is an Anti-Threshold*.

Main Surprising Result. One axiomatizations for IΣ_0 is a Herbrandized Threshold — and oddly another is an Anti-Threshold.

A Quiet Question that Needs To Be Seriously Asked ?

How do two equivalent axiomatizations for IΣ_0 Manage to have fully opposite Herbrandized Threshold properties ???

Answer. The statement $\alpha \cong \beta$ merely means that α and β prove the same set of theorems. *It does not indicate* that they can prove the formal statement " $\alpha \cong \beta$ ".

Our result about IΣ_0's puzzling Herbrandized Threshold and Anti-Threshold properties will involve constructing two equivalent systems, α and β , unable to prove that " $\alpha \cong \beta$ ".

Definition 7 Bounded Quantifiers $\forall\, x \leq T$ and $\exists\, x\ \leq T$ *are called Restricted* when T consist of *one single variable only.*

i.e. function symbols are not allowed in T

Definition 8. A formula is called Δ_0^R iff it is a Δ_0 formula — all of whose bounded quantifiers are so restricted.

Definition 9. Let us recall that Ax-2 was defined as the axiomatization of IΣ_0 that consisted of the union of axiom system Q with the following induction scheme *for all* Δ_0 *formula* $\phi(x,y)$ *:*

$$\forall x\, \forall z \,\{\ \{\ \phi(x,0)\ \wedge\ \forall y \leq z\, [\, \phi(x,y) \implies \phi(x,y+1)\ ]\ \} \\ \implies\ \forall y \leq z\ \ \phi(x,y)\quad \}$$

The axiom system Ind^R will have an identical definition as Ax-2 except it will use the preceding induction scheme only when $\phi(x,y)$ is Δ_0^R.

Theorem 1. Exists set of Π_1^R sentences, called Trivial-R, where Ax-2 $\cong$ Ind^R+Trivial-R.

Below again is Ax-2's Δ_0 induction axiom:

$$\forall x \, \forall z \, \{ \, \{ \, \phi(x,0) \; \wedge \; \forall y \le z \, [\, \phi(x,y) \implies \phi(x,y+1) \,] \, \} \implies \forall y \le z \;\; \phi(x,y) \quad \}$$

The axiom system Ind^R will have an identical definition as Ax-2 except it will use the preceding induction scheme only when $\phi(x,y)$ is Δ_0^R.

Theorem 1. Exists set of Π_1^R sentences, called Trivial-R, where Ax-2 $\cong$ Ind^R+Trivial-R.

Proof Sketch: In one direction this equality holds because each induction axiom of Ind^R is an induction axiom of Ax-2. In other direction, equality holds because each induction axiom of Ax-2 with n logical symbols has a proof from Ind^R+Trivial-R with length $O(\, 2^n \,)$.

Clarifying Comment. This $O(\, 2^n \,)$ expansion in proof length is the reason we are able to construct two equivalent axiom systems, one of which will be a *threshold* for the Herbrandized version of the Second Incompleteness Theorem — *and the other an anti-threshold !*

List of Main Theorems

Theorem 1. There exists a formal set of Π_1^R sentences, called Trivial-R such that :

$$\text{Ax-2} \cong \text{Ind}^R + \text{Trivial-R}$$

Theorem 2. Let Ax-3 denote the system Ind^R+Trivial-R. This system Ax-3 is an *Anti-Threshold* relative to the Herbrandized version of the Second Incompleteness Theorem.

Theorem 3. In contrast, Ax-1 and Ax-2 are *Thresholds* for the Herbrandized version of the Second Incompleteness Theorem.

Intuition behind contrast between Theorem 2 and Theorem 3 on next slide.

Difference Between Δ_0 and Δ_0^R Formulae:

Let $\Psi_K = \exists\ y \leq 2^{2^K}\ \phi(y)$. It has following two properties:

- Ψ_K's formal encoding has a 2^K length when it is written as a Δ_0^R formula (because 2^k digits are needed to encode " 2^{2^K} ")
- *In contrast,* Ψ_K's encoding has an $O(\ K\ )$ length when it is written as a Δ_0^R formula because it can be encoded as:

$$\exists x_0 \leq 2\ \ \exists x_1 \leq (x_0)^2\ \ \exists x_2 \leq (x_1)^2\ \ \ldots$$

$$\exists x_k \leq (x_{k-1})^2\ \ \exists y \leq (x_k)\quad \phi(y).$$

This difference in sentence lengths explains intuition why Ax-2 and Ax-3 definitions of IΣ_0 have opposite incompleteness properties *despite the fact they prove the same theorems !*

Revisiting our List of Main Theorems

Theorem 1. There exists a formal set of Π_1^R sentences, called Trivial-R such that :

$$\text{Ax-2} \cong \text{Ind}^R + \text{Trivial-R}$$

Theorem 2. Let Ax-3 denote the system Ind^R+Trivial-R. This system Ax-3 is an *Anti-Threshold* relative to the Herbrandized versions of the Second Incompleteness Theorem.

Theorem 3. In contrast, Ax-1 and Ax-2 are *Thresholds* for the Herbrandized version of the Second Incompleteness Theorem.

Intuitive reason for contrast between Theorem 2 and Theorem 3 is the difference in length for encoding Ψ_k as a Δ_0 and Δ_0^R formula.

$$\Psi_K \quad = \quad \exists\, y \,\leq 2^{2^K}\; \phi(y)$$

Main Surprising Result. One axiomatizations of $\mathsf{I}\Sigma_0$ is a Herbrandized Threshold — and another is an Anti-Threshold.

A Quiet Question that Needs To Be Seriously Asked ?

How do two equivalent axiomatizations for $\mathsf{I}\Sigma_0$ manage to have *Fully Opposite* Threshold properties ???

Answer. The statement $\alpha \cong \beta$ merely means that α and β prove the same set of theorems. *It does not indicate* that they can prove the formal statement " $\alpha \cong \beta$ ".

Our result about $\mathsf{I}\Sigma_0$'s puzzling Herbrandized Threshold and Anti-Threshold properties involves constructing two equivalent systems, α and β , unable to prove that " $\alpha \cong \beta$ ".

Concluding Remark :

Generalizations of Gödel's Second Incompleteness Theorem are much more important than its occasional boundary-case exceptions. However in a context where the Incompleteness Theorem has been called the centennial theorem of 20-th century mathematics, the latter topic should also be explored to help sharpen our knowledge of the exact meaning of Gödel's result.

Concluding Joke :

My original 1993 paper on this topic represented a perhaps 0.1 % Re-Interpretation of Gödel's Centennial Incompleteness Theorem. The combined new work in the last 12 years is perhaps a Factor-30 Improvement over the initial work

... i.e. a perhaps 3 % Re-Interpretation of Gödel's Centennial Theorem.

Serious Remark : If this combined research does represent a "3 % Re-Interpretation" of the meaning Gödel's Centennial Theorem, then it is a *serious, albeit limited, result.*