WoLLIC 2005

Problem description

We have already one correct answer (someone that you all know has solved the problem in 1940!). We have also a much better formalization of the problem, including the determination of upper bounds for the number of theories. Below a more detailed description of the problem and the solution of the two propositional symbols theories.

Let $\mathcal{L}_4$ be the set of all non tautological, non contradictory logical theories that can be constructed using the usual classical logical connectives $(\wedge, \vee, \rightarrow, \neg)$ and the set of propositional symbols $P = \{p_1, p_2, p_3, p_4\}$.

Consider the equivalence relation $\Leftrightarrow_{W05}$, named from WoLLIC'05, among elements of $\mathcal{L}_4$:

$$t \Leftrightarrow_{W05} t' \text{ iff } [\![t]\!] = [\![t']\!]$$

where $[\![t]\!]$ is the set of all models of t, according to the usual classical propositional logic semantics, modulo the propositional symbol names and signals, i.e., the set of all distinct theories (or Boolean formulas) with respect to the group of permutations and complementations.

This last restriction intends to capture the idea of a logical theory as a relation among anonymous logical variables, characterized only by the logical properties of the syntactical expressions that represent it. This restriction can be formalized as follows: given $t, t' \in \mathcal{L}_4$, i.e., the syntactical representations of two logical theories are equivalent *modulo the propositional symbol names and signals*, if they can be transformed among them simply by renaming their propositional symbols, possibly by a negated value. Such transformations can be formalized as sequences of applications of two syntactical transformation functions:

- *Exchange* $(\mathcal{X} : \mathcal{L}_4 \times P \times P \rightarrow \mathcal{L}_4)$: given a theory in $\mathcal{L}_4$ and two propositional symbols, exchange the identity of the symbols in the given theory.

- *Flip* ($\mathcal{F} : \mathcal{L}_4 \times P \rightarrow \mathcal{L}_4$) : given a theory in $\mathcal{L}_4$ and one propositional symbol, negate all occurrences of the associated literals in the theory.

Although the theories $t \in \mathcal{L}_4$ are syntactic objects, the definition of the relation $\Leftrightarrow_{W05}$ has also a semantic part. Therefore, given the set $P = \{p_1, p_2, p_3, p_4\}$ of propositional symbols, we have, for instance:

- Theories that are is the same equivalence class because they are syntactically equivalent, i.e., they differ only by the propositional symbol names and signals:

$$(p_1 \vee \neg p_2) \wedge (p_2 \vee p_3) \Leftrightarrow_{W05} (\neg p_2 \vee p_1) \wedge (\neg p_1 \vee p_3)$$

$$p_1 \wedge p_3 \Leftrightarrow_{W05} \neg p_2 \wedge \neg p_4$$

$$p_1 \Leftrightarrow_{W05} \neg p_4$$

- Theories that are is the same equivalence class because they are semantically equivalent, i.e., they can be transformed into one another through valid inference rules (that preserve the models):

$$p_1 \wedge (p_1 \rightarrow p_3) \Leftrightarrow_{W05} p_1 \wedge p_3 \text{ (modus ponens)}$$

$$p_1 \wedge (p_1 \vee p_2) \Leftrightarrow_{W05} p_1 \text{ (subsumption)}$$

> **Question:** what is the cardinality of the partition of $\mathcal{L}_4$ generated by the above equivalence relation, more formally, given:
>
> $$T = \{[t] \mid t \Leftrightarrow_{W05} t', t' \in \mathcal{L}_4\}$$
>
> what is $\mid T \mid$? Or in words: what is the maximal number of "structurally different" theories that can be built with at most four propositional symbols?

The solution for two propositional symbols – $P = \{p_1, p_2\}$ – is as follows. There are 4 possible models[1]:

$$\{\langle \neg p_1, \neg p_2 \rangle, \langle \neg p_1, p_2 \rangle, \langle p_1, \neg p_2 \rangle, \langle p_1, p_2 \rangle\}$$

[1] To simplify the notation, a model is noted as a set of n literals, where n is the number of propositional symbols in P, such that $\langle \psi_1, \ldots, \psi_i, \ldots, \psi_n \rangle$ represents the assignment $\epsilon(p_i) = true$ if $\psi_i = p_i$ or $\epsilon(p_i) = false$ if $\psi_i = \neg p_i$, and $\epsilon : P \rightarrow \{true, false\}$ is the semantic function that maps propositional symbols into truth values.

All non tautological, non contradictory logical theories must be true in a subset of these 4 models, therefore the maximal number of theories is given by:[2]

$$C_1^4 + C_2^4 + C_3^4 = 4 + 6 + 4 = 14$$

Careful analysis (and how to proceed with this analysis is what the question is all about!) shows that the 4 theories with one model reduce to 1, modulo $\Leftrightarrow_{W05}$, the 6 theories with 2 models reduce to 2 and the 4 with 3 models, symmetrically to those with 1 one model, also reduce to 1, given a total number of 4 equivalence classes for the relation $\Leftrightarrow_{W05}$.

Solution: representing each model of a theory as a conjunction of literals and taking their disjunction is one possible (disjunctive normal form) syntactical representation of the theory.

- The 4 theories that have one model can be represented by:

  $$p_1 \wedge p_2 \qquad \neg p_1 \wedge p_2 \qquad p_1 \wedge \neg p_2 \qquad \neg p_1 \wedge \neg p_2$$

  and they are all syntactically equivalent w.r.t. $\Leftrightarrow_{W05}$, e.g., the first can be transformed into the second by the $Flip(p_1)$ operation.[3]

- The 6 theories that have two models can be represented by:

  $$\begin{array}{ll} (p_1 \wedge p_2) \vee (p_1 \wedge \neg p_2) & (p_1 \wedge p_2) \vee (\neg p_1 \wedge \neg p_2) \\ (p_1 \wedge p_2) \vee (\neg p_1 \wedge p_2) & (p_1 \wedge \neg p_2) \vee (\neg p_1 \wedge p_2) \\ (p_1 \wedge \neg p_2) \vee (\neg p_1 \wedge \neg p_2) & \\ (\neg p_1 \wedge p_2) \vee (\neg p_1 \wedge \neg p_2) & \end{array}$$

  Theories in the same column are equivalent w.r.t. $\Leftrightarrow_{W05}$, e.g., consider those in the left column, the first line can be transformed into the second by the $Exchange(p_1, p_2)$ operation and into the fourth by the $Flip(p_1)$ operation. In the right column, the two lines can be transformed into each other using either $Flip(p_1)$ or $Flip(p_2)$ operations. But *no* sequence of *Flip* and *Exchange* operations can transform theories in one column into theories of the other column.

[2] C_m^n is the number of combinations of n elements taken m at a time.

[3] We have suppressed the theory argument of the *Flip* and *Exchange* operations to simplify the notation.

- The 4 theories that have three models can be represented by:

$$
\begin{array}{l}
(p_1 \wedge p_2) \vee (p_1 \wedge \neg p_2) \vee (\neg p_1 \wedge p_2) \\
(p_1 \wedge p_2) \vee (p_1 \wedge \neg p_2) \vee (\neg p_1 \wedge \neg p_2) \\
(p_1 \wedge p_2) \vee (\neg p_1 \wedge p_2) \vee (\neg p_1 \wedge \neg p_2) \\
(p_1 \wedge \neg p_2) \vee (\neg p_1 \wedge p_2) \vee (\neg p_1 \wedge \neg p_2)
\end{array}
$$

  and they are also all syntactically equivalent w.r.t. $\Leftrightarrow_{W05}$, e.g., the second line can be transformed into the third by the $Exchange(p_1, p_2)$ operation and into the first by the $Flip(p_1)$ operation.

Therefore, there are four "structurally different" theories with two propositional symbols:

$$
\begin{array}{c}
p_1 \wedge p_2 \\
(p_1 \wedge p_2) \vee (p_1 \wedge \neg p_2) \qquad (p_1 \wedge p_2) \vee (\neg p_1 \wedge \neg p_2) \\
(p_1 \wedge p_2) \vee (p_1 \wedge \neg p_2) \vee (\neg p_1 \wedge p_2)
\end{array}
$$

and a possible (naïve) methodology to solve the problem is:

1. Find the disjunctive normal form representation of all theories in $\mathcal{L}_4$ that have 1, 2, ..., 15 models using the disjunction of their models taken as conjunctions.

2. Use the *Flip* and *Exchange* operations to find the equivalence classes w.r.t. $\Leftrightarrow_{W05}$ in each of these theory sets.

The table below summarizes the results for the 2 propositional symbols case, gives the solution for the 3 propositional symbols case and establishes an upper bound for the 4 propositional symbols case.

n		$C_1^{2^n}$	$C_2^{2^n}$	$C_3^{2^n}$	$C_4^{2^n}$	$C_5^{2^n}$	$C_6^{2^n}$	$C_7^{2^n}$	$C_8^{2^n}$	$\vdots$	$\sum$
2	*all*	4	6	4						$\vdots$	14
	w05	1	2	1						$\vdots$	4
3	*all*	8	28	56	70	56	28	8		$\vdots$	254
	w05	1	3	3	6	3	3	1		$\vdots$	20
4	*all*	16	120	560	1820	4368	8008	11440	12870	$\vdots$	65534
	w05	?	?	?	?	?	?	?	?	$\vdots$	?

Answers should be sent by e-mail to gb@das.ufsc.br (subject: WoLLIC'05 problem) before 20/07/2005. The next first 4 correct answers will receive a free copy of the WoLLIC'05 proceedings.